Capacitors As Buckets:
An Analogy In Two Parts, Sort Of

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To assist with understanding both the transient DC and steady state AC responses of capacitor circuits, we're going to use an analogy with water and buckets. We will begin with first principles, and eventually work through charging waveforms and the concept of capacitive reactance.

Part I

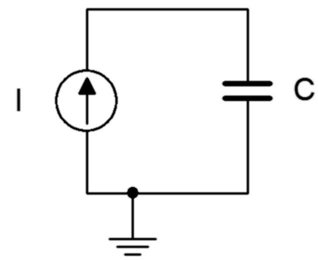
Let's begin with charge and current. Charge is a characteristic of an item. Saying that something has a charge is like saying saying that someone is charming. Electrons have a negative charge, although individually it is very small. If we gather up enough electrons (a little over six billion billion of them), we arrive at a more convenient unit of charge, the coulomb, denoted by the letter Q. Electrical current is nothing more than the movement of charge from place to place over a certain amount of time, that is, the rate of charge transfer. Current equals charge divided by time, or $I = Q/t$. If we can move one coulomb through a wire in the space of one second, we have a current of one amp.

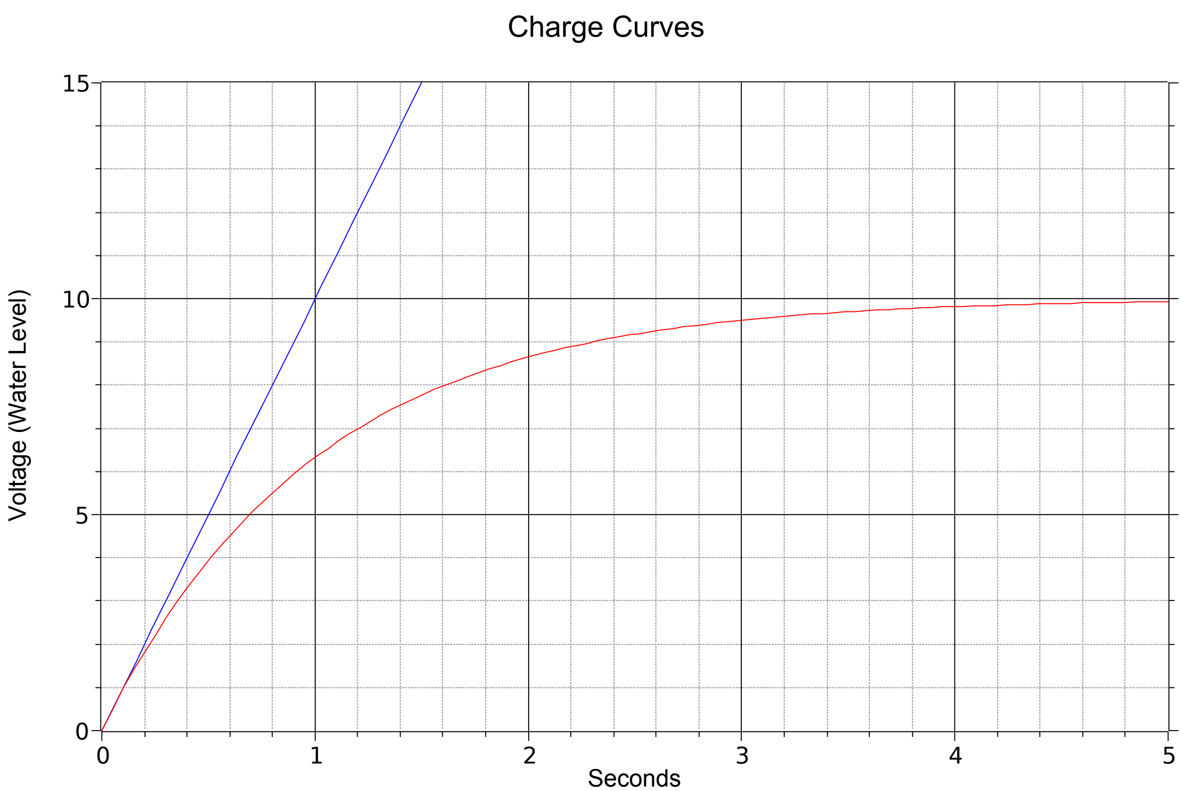
For our analogy, we shall think of water molecules like electrons. Just as we gathered up a large number of electrons to arrive at a coulomb of charge, we can gather up a large number of water molecules and give it a convenient name such as a liter (or a gallon). Thus, the rate of flow of water is like the aptly named electrical current. One might be measured in liters per minute while the other is measured in amps (coulombs per second).

In this analogy, a constant current source would be like a faucet delivering a certain number of liters of water per minute. We might want to store this water in some container temporarily, such as a bucket. Similarly, we might want to store electric charge temporarily. The device that does this is the *capacitor*. The capacitance is analogous to the *capacity* of the bucket, which is proportional to the diameter of the bucket (i.e., a large diameter bucket can hold a lot of water, just like a large value capacitor can store a lot of charge). The amount of water in the bucket is analogous to the amount of charge stored in a capacitor, and the water level in the bucket is analogous to the voltage across the capacitor.

The fundamental equation that describes the operation of a capacitor is $i = C\, dv/dt$. The dv/dt term is called a *derivative*. It is just a value or expression that describes how the voltage changes with time. In other words, the current through the capacitor is equal to the capacitance times the rate of change of voltage across the capacitor (i.e., its slope). For our purposes, it is more useful to rewrite this as $dv/dt = i/C$, or the rate of change of capacitor voltage is equal to the capacitor's current divided by the capacitance. Thus, if we want the voltage to change rapidly (i.e., have a very steep slope), we need either a large current source or a small capacitance. It is useful to note that **the voltage across a capacitor cannot change instantaneously**. An instantaneous change means the slope of the voltage would be perfectly vertical, and thus infinite. According to this equation, the only way we could get an infinite slope is if the current source was also infinite. In the real world, there is no such thing as an infinitely large current source, therefore the voltage across a capacitor cannot change instantaneously.

OK, so if we connect a fixed DC current source to a capacitor, this equation tells us that the voltage across the capacitor will begin to rise at a constant rate. For example, if 1 mA of current feeds a 100 μF capacitor, the voltage slope is 1 mA/100 μF or 10 volts per second. If we measure the capacitor voltage one second after turning on the power, it should be 10 volts. Remember, a 1 mA current means that 1 millicoulomb of charge is being transferred each second. Therefore the charge transferred to the capacitor would be 1 mC. If we wait another second, the voltage should rise another 10 volts for 20 volts total. At the end of one minute there should be 600 volts across the capacitor. It also means that 60 millicoulombs of charge would be stored in the capacitor. This trend is depicted by the straight blue line in the plot below for the first 1.5 seconds, reaching 15 volts.

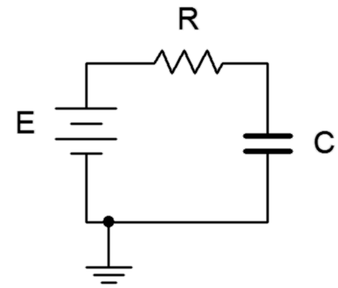




If we turned off the current source after one minute, an ideal capacitor would hold this charge unchanged, meaning the voltage would stay at 600 volts forever. Further, if we then flipped the polarity of the current source, it would start draining the stored charge, and voltage would begin to decrease.

So what does all of this have to do with buckets? Remember, the capacitor is modeled as a bucket and the flow of water is modeling the flow of current. If we place the bucket under the faucet and turn it on, what happens? The water level in the bucket begins to rise. Remember, the water level is analogous to the capacitor's voltage. The longer we wait, the higher the water level gets, just like the capacitor's voltage. Also, we can see that a higher water flow rate will raise the water level faster, just like a higher electrical current raises the capacitor's voltage faster. Further, if we use a small diameter bucket, the water level will rise faster as well, just like the way a small capacitance can be charged faster. If we turn off the faucet, the amount of water should stay constant, and thus the level stays put, just like the capacitor's voltage. In the real world, the bucket might be leaky, and the water will slowly dribble out with the water level slowly decreasing. This is precisely what is meant by a “leaky” capacitor, that is, the charge slowly bleeds off through some internal resistance, and the voltage slowly decays back to zero. Finally, if we were able to turn the faucet handle backward so that it turned into a suction pump, we could remove the water from the bucket just like reversing the polarity of the current source removed the stored charge on the capacitor.

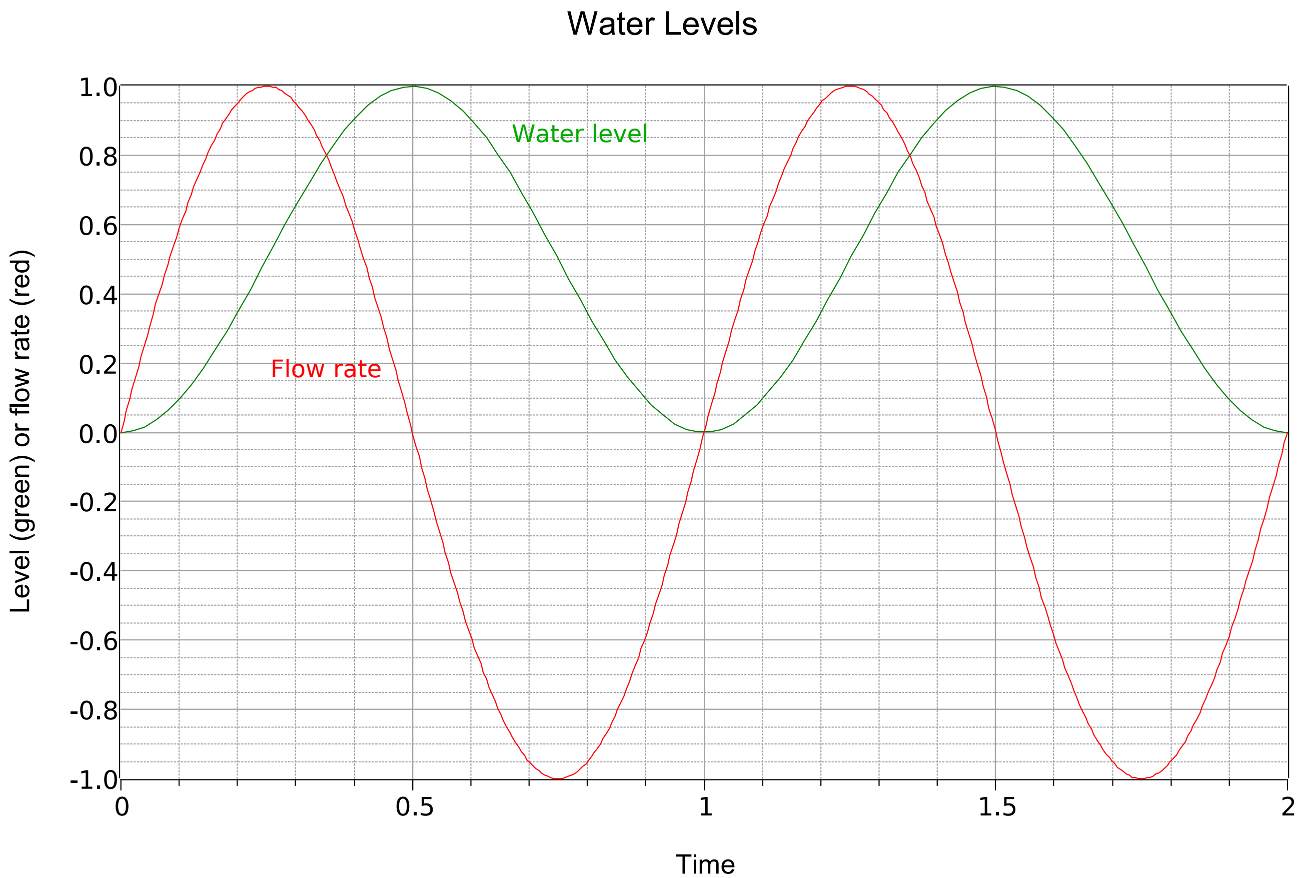
A fair question at this point is, “What happens if we use a voltage source and a resistor instead of a current source?” As soon as power is applied, all of the voltage would drop across the resistor because the voltage across the capacitor can't change instantaneously. This creates a current which starts charging the capacitor. Due to Kirchhoff's Voltage Law, as the capacitor voltage starts to rise, the voltage across the resistor starts to decrease because these two voltages must add up to the voltage source. As the resistor voltage decreases, the resulting current also decreases due to Ohm's Law. A decreased current means that the rate of change of voltage across the capacitor begins to slow. The larger the capacitor voltage, the smaller the current, and the slower the capacitor voltage changes. Once the capacitor voltage gets close to the power supply voltage, the resulting current is so small that the rate of change of capacitor voltage nearly slows to a halt. Using our bucket analogy, it's as if we have suspended the bucket with a rope attached to pulleys that connect to the back of the faucet's handle: The more water that's in the bucket, the heavier it is, and the more it pulls on the rope which pulls back on the handle, slowly turning off the water. If we use the same 100 μF capacitor as before and add a 10 volt source in series with a 10 kΩ resistor, we'll get the same initial current of 1 mA, and hence, the same initial slope. The voltage (water level) will continually increase until it reaches the maximum, but it slows as it does so. This is depicted by the red curve in the plot above.



Part II

Now what happens if instead of using a constant current (DC at a fixed amplitude), we use a current that keeps changing amplitude (AC)? This would be like throttling the handle of the faucet from low flow to high flow and back, repeatedly. Indeed, if we want to make this analogy a little more accurate, we would include the ability of our faucet to go into “suction mode”. We could continually move the handle smoothly back and forth between normal and suction modes, producing increasing and decreasing intensities of water flow over and over, thus repeatedly filling and draining the bucket.

Here's how it works: At the start, the faucet is off and the bucket is empty. As we turn the handle, the water flow begins to increase and the water level in the bucket begins to rise. Remember, the water flow corresponds to electrical current and the height of the water in the bucket corresponds to the capacitor's voltage. As we crank up the handle, the water goes from a trickle to a blast, and the level in the bucket begins to increase slowly at first, and then rapidly. Once we max out the handle, we begin to turn it back to the off position. During this time, the water level is still rising, it's just not rising as fast as it was initially. In fact, the water reaches maximum height the instant the faucet handle hits “off”. If we now turn the handle to “suction mode”, the water level starts to decrease. The water level will hit zero only after the handle has gone to max suction and back to “off”. At this point we'll simply repeat the process, over and over. If we look at the water level curve, we notice that it lags the flow of water. That is, the maximum water level only occurs *after* the flow of water has peaked and then gone back to zero. By analogy then, the capacitor voltage will lag behind the AC current. In fact, it lags by precisely one quarter of a cycle, or 90 degrees. This is illustrated in the following plot.



To continue, what happens if we turn the handle back and forth much faster than we did originally? Basically, during the "on" phase, less water will have entered the bucket, leaving a lower water level. Thus by analogy, the higher the frequency of the current, the lower the voltage across the capacitor. Similarly, the larger the bucket, the lower the water level will be, which implies that the larger the capacitor, the smaller the voltage.

An important observation involves how the voltage *reacts* to the AC current. We just discovered that the voltage is inversely proportional to both the capacitance value and the frequency. We refer to this characteristic as the *capacitive reactance*, or X_C. It is equal to $1/(2\pi fC)$. Because this is a ratio of voltage to current, we give it units of ohms, however, the time lag must be included, so we say that the reactance is 90 degrees lagging, or $-j\ X$ ohms.

Part III (Bonus)

Of course, we could just look at this from a purely mathematical standpoint and forget all of the visuals with water and buckets and so forth. The capacitive reactance is defined as $X_C = v_C/i_C$. If the voltage is a sine wave, then $v(t) = V_P \sin 2\pi ft$. Recalling that $i = C\ dv/dt$, the current would be the derivative of this times the capacitance, which is $CV_P 2\pi f \cos 2\pi ft$. Dividing the voltage by the current gives us $V_P \sin 2\pi ft\ /\ CV_P 2\pi f \cos 2\pi ft$. The peak voltage terms cancel and the cosine can be rewritten as a sine with a 90 degree shift. This yields $\sin 2\pi ft\ /\ C2\pi f \sin 2\pi ft{+}90°$, which can be further simplified to $X_C = 1\ /\ 2\pi fC$ at $-90°$, or $-j\ 1\ /\ 2\pi fC$.