Methods of Solution of Linear Simultaneous Equations

DC Circuits

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Several circuit analysis methods such as branch current analysis, mesh analysis and nodal analysis, yield a set of linear simultaneous equations. There will be as many equations as there are unknowns. For example, a particular circuit might yield three equations with three unknown currents (often referred to as a "3 by 3" for the matrix it creates). There are several techniques that may be used to solve this system of equations. The methods include graphical, substitution, Guass-Jordan elimination and determinants (determinants may be solved via Cramer's Rule/Sarrus' Rule or via expansion by minors).

Graphical

Graphical solutions involve plotting the individual equations on graph paper. The location of where the lines cross is the solution to the system (i.e., values that satisfy all of the equations). This technique will not be discussed further because it is only practical for two unknowns. It would be very difficult to draw something like a four dimensional graph for four equations with four unknowns!

Substitution

The idea here is to write one of the equations in terms of one of the unknowns and then substitute this back into one of the other equations resulting in a simplified version. This process is iterated for as many unknowns as the system includes. Take for example the following 2x2:

$$10 = 20I_1 + 8I_2$$

$$2 = 8I_1 + 4I_2$$

Solve the second equation for I_2.

$$2 = 8I_1 + 4I_2$$

$$4I_2 = 2 - 8I_1$$

$$I_2 = .5 - 2I_1$$

Substitute this back into the first equation and expand/simplify/solve.

$$10 = 20I_1 + 8I_2$$

$$10 = 20I_1 + 8(.5 - 2I_1)$$

$$10 = 20I_1 + 4 - 16I_1$$

$$10 = 4I_1 + 4$$

$$I_1 = 1.5$$

Finally, substitute this value back into one of the two original equations to determine I_2.

$$2 = 8I_1 + 4I_2$$

$$2 = 8(1.5) + 4I_2$$

$$2 = 12 + 4I_2$$

$$4I_2 = -10$$

$$I_2 = -2.5$$

For a 3x3, this process is iterated as follows: Equation 2 would be solved for I_3 and this would be substituted back into equation 1 yielding a new equation (let's call it A) with only I_1 and I_2 terms. Similarly, Equation 3 would be solved for I_3 and this would be substituted back into equation 2 yielding a new equation (let's call it B) with only I_1 and I_2 terms. Equations A and B now make a 2x2 with I_1 and I_2 as the unknowns and can be solved as outlined above. This would yield values for I_1 and I_2 which could then be substituted into one of the three original equations to obtain I_3. While the substitution method is perfectly valid for an arbitrarily sized system, it proves cumbersome as the system gets larger.

Gauss-Jordan Elimination

In some respects, Gauss-Jordan is similar to substitution but it tends to involve less overhead for larger systems and thus is generally preferred. This method involves multiplying one equation by a constant such that when it is subtracted from another equation, one of the unknown terms disappears. The process is then iterated for as many unknowns exist in the system. Using the same example from before:

$$10 = 20I_1 + 8I_2$$

$$2 = 8I_1 + 4I_2$$

Multiply the second equation by the ratio of the coefficients for I_2 (8/4 = 2).

$$2 = 8I_1 + 4I_2$$

$$4 = 16I_1 + 8I_2$$

Subtract this new equation from the first equation. The I_2 terms will cancel leaving just I_1.

$$10 = 20I_1 + 8I_2$$

$$4 = 16I_1 + 8I_2$$

$$6 = 4I_1$$

$$I_1 = 1.5$$

Substitute this result back into one of the original equations to obtain I_2. For a 3x3, iterate as follows: Using equations 1 and 2, multiply equation 2 by the ratio of the coefficients for I_3. Subtract this equation from equation 1 to generate a new equation (let's call it equation A) that only has I_1 and I_2 as unknowns. Using equations 2 and 3, multiply equation 3 by the ratio of the coefficients for I_3. Subtract this equation from equation 2 to generate a new equation (let's call it equation B) that only has I_1 and I_2 as unknowns. . Equations A and B now make a 2x2 with I_1 and I_2 as the unknowns and can be solved as outlined previously. This would yield values for I_1 and I_2 which could then be substituted into one of the three original equations to obtain I_3. Like the substitution method, Gauss-Jordan grows rapidly as the system size increases. The process tends to be formulaic though, and thus easier to handle.

Determinants

Determinants revolve around the concept of a matrix which itself is little more than an ordered collection of coefficients and/or constants. It is imperative that the unknowns be in the same order in each equation (i.e., I_1 ascending to I_X left to right) A simple coefficient matrix for the original 2x2 example is:

```
20  8

 8  4
```

The resultant value (properly referred to as the *determinant*) for a 2x2 matrix such as this may be solved using **Sarrus' Rule**: Simply multiply the two values along the upper right-lower left diagonal and then subtract that product from the product of the two values found along on the upper left-lower right diagonal. In this example that's:

```
20*4 – 8*8 = 16
```

A solution involves dividing one determinant by another determinant (Cramer's Rule). That is, each matrix is solved for its resultant value and then these two values are divided to determine the final answer. One of these matrices will be the coefficient matrix just discussed. This will be placed in the denominator. The numerator matrix is a modified version of the basic coefficient matrix. It is created by replacing one column of coefficients with the constant values from the original system of equations. For example, the numerator matrix used to find I_1 would replace the first column (the I_1 coefficients 20 and 8) with the constants 10 and 2:

```
10  8

 2  4
```

The resultant value is 40-16 or 24.

Similarly, the numerator matrix for I_2 would replace the I_2 coefficients in the second column (8 and 4) with the constants 10 and 2:

```
20 10

 8  2
```

The resultant value is 40-80 or -40. To find any particular unknown, simply divide the modified matrix by the basic coefficient matrix.

```
            10  8

             2  4

I1 = --------

            20  8

             8  4
```

$$I_1 = \frac{24}{16}$$

$$I_1 = 1.5$$

In like fashion I_2 is found:

$$I_2 = \frac{\begin{vmatrix} 20 & 10 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 20 & 8 \\ 8 & 4 \end{vmatrix}}$$

$$I_2 = \frac{-40}{16}$$

$$I_2 = -2.5$$

Sarrus' Rule may also be used with a 3x3 matrix. This is achieved by extending the matrix. Fourth and fifth columns are added to the right of the 3x3 matrix by simply making copies of the first two columns. This creates three right to left diagonals with three values each and three left to right diagonals with three values each. The three values along each diagonal are multiplied together. The three right to left products are then subtracted from the sum of the three left to right products yielding a single resultant value (the determinant). To create the modified numerator matrix, replace the coefficient column of interest with the constant terms and then replicate columns one and two. For example, given these three equations:

$$10 = 20I_1 + 8I_2 + 3I_3$$

$$2 = 8I_1 + 4I_2 + 5I_3$$

$$7 = 3I_1 + 5I_2 + 6I_3$$

The basic coefficient matrix (i.e., denominator) is:

$$\begin{matrix} 20 & 8 & 3 \\ 8 & 4 & 5 \\ 3 & 5 & 6 \end{matrix}$$

The extended matrix is:

$$\begin{matrix} 20 & 8 & 3 & 20 & 8 \\ 8 & 4 & 5 & 8 & 4 \\ 3 & 5 & 6 & 3 & 5 \end{matrix}$$

The result is:

```
20*4*6 + 8*5*3 + 3*8*5 - 3*4*3 - 20*5*5 - 8*8*6
```

Sarrus' Rule does not work beyond 3x3.

Expansion by Minors is another method that may be used to generate a determinant solution. This involves breaking the matrix into a series of smaller matrices (minors) that are combined using row-column coefficients. The position of these coefficients will also indicate whether the determinant of any particular sub-matrix is added or subtracted to the total.

The first step is to establish a single row or column from which to derive the coefficients. This can be any horizontal row or vertical column (no diagonals). Each element of the chosen row or column determines the associated minor (essentially, that which is left over). Consider the 3x3 used previously:

```
20  8  3

 8  4  5

 3  5  6
```

Choosing the top row yields coefficients of 20, 8 and 3. For each of these, blot out its row and column and see what is left. This leaves three 2x2 matrices, one for each coefficient. Multiply each coefficient by the determinant of its 2x2 matrix. To determine whether this result is added or subtracted to the others, the sign may be found using the following map for the coefficients:

```
+  -  +  - etc.

-  +  -  + etc.

+  -  +  - etc.
```

The origin in the upper left is positive and the signs continually alternate across from it and down from it. The result using the top row for the coefficients is found thus (the 2x2 matrices are **bold red** for clarity):

```
     4 5        8 5        8 4

20 * 5 6 - 8 * 3 6 + 3 * 3 5
```

If the second column was used instead (8, 4, 5), the result is found like so:

```
     8 5       20 3       20 3

-8 * 3 6 + 4 * 3 6 - 5 * 8 5
```

In closing, whichever method is used, always look for null coefficient terms (that is, places in the equations and matrices where the coefficients are zero). Smart use of these can considerably simplify the computations as there are few mathematical operations easier than multiplying by zero. For example, if a particular row of a matrix contains a few zeros, that would be a good candidate for the coefficient row when using expansion by minors because some 2x2 minors need not be computed (they will just be multiplied by the zero coefficient).