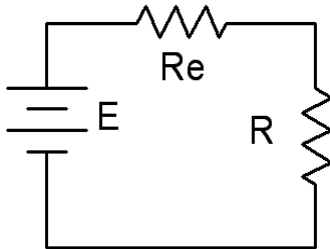


DC Maximum Power Transfer Theorem Proof

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Given a simple voltage source with internal resistance, a useful question to ask is “What value of load resistance will yield the maximum amount of power in the load?” While it is not true that maximizing load power is a goal of all circuit designs, it is a goal of a portion of them and thus worth a closer look. Consider the basic circuit below with load resistance R:



We would like to describe the load power in terms of the load resistance. To make the job easier, we may normalize the voltage source E to 1 volt and the source resistance R_e to 1 Ohm. By doing this, R also becomes a normalized value, that is, it no longer represents a simple resistance value but rather represents a ratio in comparison to R_e . In this way the analysis will work for any set of source values. Note that the value of E will equally scale the power in both R_e and R, so a precise value is not needed, and thus, we may as well chose 1 volt for convenience.

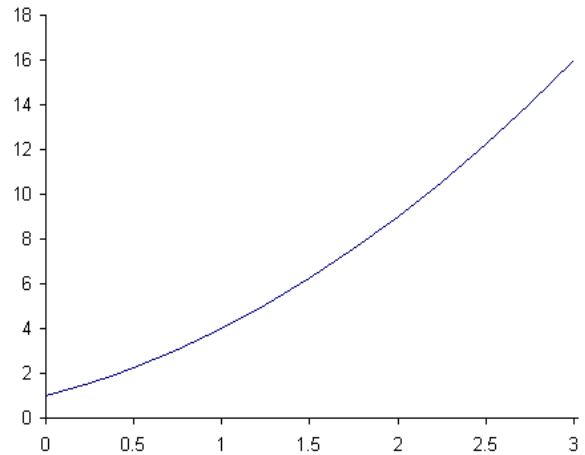
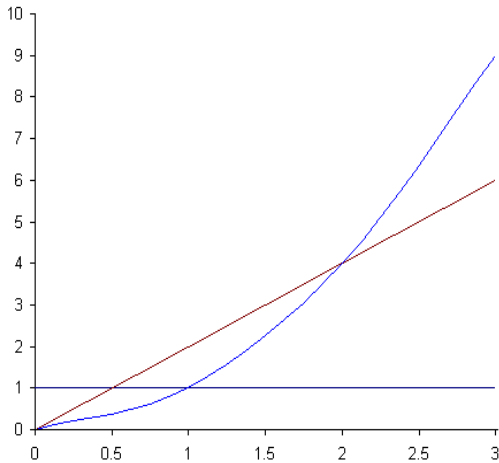
The power in the load can be determined by using I^2R where $I=E/(R_e+R)$. Using our normalized values $I=1/(1+R)$ and thus the load power is:

$$P = \left(\frac{1}{1+R} \right)^2 R \quad \text{or after expanding, } P = R/(R^2+2R+1)$$

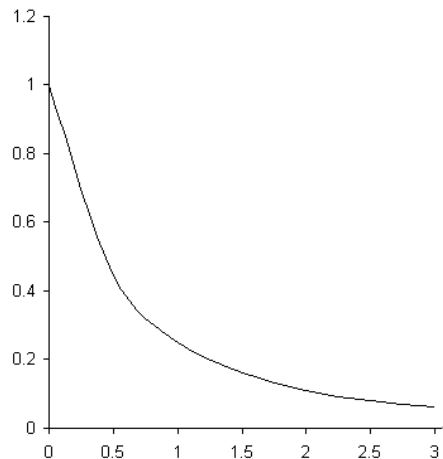
We now have an equation that describes the load power in terms of the load resistance. Before we go any further, take a look at what this equation tells you, in general. It is obvious that maximum power will not occur at the extremes. If $R=0$ or $R=\infty$ (i.e., shorted or opened load) the load power is zero. While shorting the load will yield the maximum load current, it will also yield zero load voltage and thus no power. Similarly, opening the load will yield maximum load voltage but it will also yield zero load current, and again, no load power. To find the precise value that produces the maximum load power, the proof can be divided into two major portions. The first involves graphing the function and the second requires differential calculus to solve for a precise value.

Graphing the Power Function $P = R/(R^2+2R+1)$

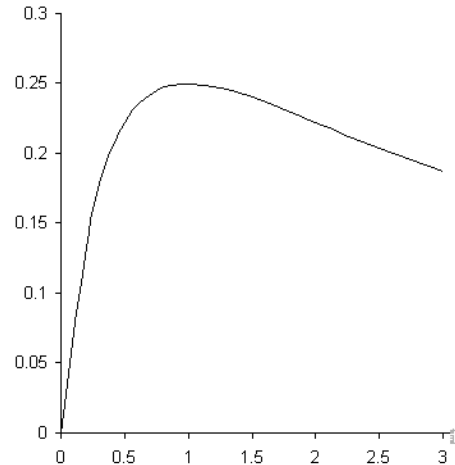
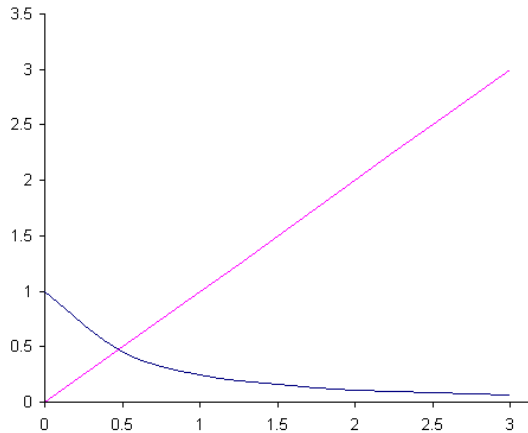
This may be done in parts, looking at the contribution of each term, and then combining to form the final result. First, consider the denominator R^2+2R+1 . It consists of three segments. The most simple is the horizontal line at +1. The $2R$ term creates a straight line with slope 2 starting at the origin. The R^2 term creates a simple curve with increasing slope that starts at the origin, crosses the horizontal +1 line at $R=1$ and also crosses the $2R$ line at $R=2$. These items are drawn individually and then added together, below:



Of course, we must remember that we really want $1/(R^2+2R+1)$ and so we plot the reciprocal of the combo:



Finally, consider that the numerator consists of the single term R , and thus is plotted as a straight line with slope 1 starting at the origin (similar to the $2R$ line above). Combining this with the denominator curve yields the final curve (drawn separately and then multiplied together, below):



An examination of an accurate rendering of this curve will show that the peak occurs at $R=1$. In other words, **maximum load power will be obtained when the load resistance is equal to the voltage source's internal resistance**. You can try substituting a few values around the peak to verify this. For example, given $E=1$ and $R_e=1$, calculate P for $R=.5$, 1 , and 2 :

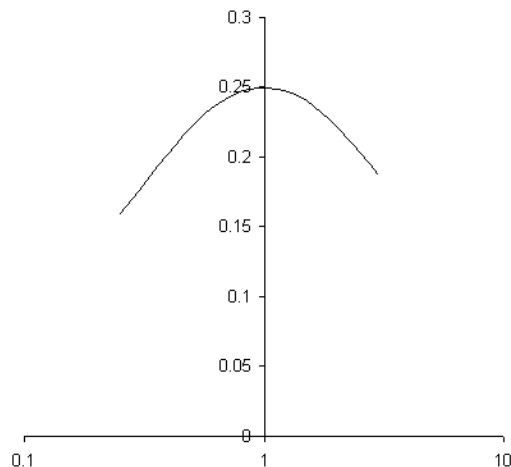
$$P_{.5} = .5 / (.5^2 + 2 \cdot .5 + 1) = 2/9$$

$$P_1 = 1 / (1^2 + 2 \cdot 1 + 1) = 2/8 = 1/4$$

$$P_2 = 2 / (2^2 + 2 \cdot 2 + 1) = 2/9$$

You can try this with extremely small variations such as $R=.9999$ and $R=1.0001$ and you will not achieve a value equal to or greater than $1/4$ Watt. It should also be noted that in the maximum power case, because $R_e=R$, the load power and internal power are the same value, and thus the load efficiency is only 50% (i.e., only half of all generated power goes to the load with the other half being wasted internally). Sometimes we favor efficiency over maximal load power.

One final item: If we plot the completed power curve using a logarithmic horizontal axis and also shift the vertical axis to unity, the peak is more apparent and the curve is symmetrical in shape rather than lopsided. This reinforces the idea that the *ratio* of R to R_e is what matters.



Refining the Maximal Value of $P = R/(R^2+2R+1)$

While the graphing technique leads to a proper answer, it is incomplete. For the ultimate value we can use a little differential calculus. Examining the curve of the function indicates a single peak, and thus we may find the corresponding value by taking the first derivative of the function, setting it to zero (i.e., find the point where the slope goes to zero), and solving for R. There are three basic techniques for determining the derivative of the equation: using quotient rule, using product rule, and using chain rule. All three will be examined for the sake of completeness.

1. Quotient Rule

The quotient rule states that the derivative of a quotient is

$$\frac{f(x)'}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

This is the most obvious technique to apply (although perhaps not the quickest, as we shall see)

$f(x)$ is R and therefore $f'(x)$ is 1

$g(x)$ is R^2+2R+1 and therefore $g'(x)$ is $2R+2$

Note that for simplicity, $g(x)$ may also be written as $(R+1)^2$

Plugging these values into the quotient formula yields

$$\frac{dP}{dR} = \frac{(R+1)^2 \cdot 1 - R(2R+2)}{(R+1)^4}$$

This may be simplified to

$$\frac{dP}{dR} = \frac{1 - R^2}{(R+1)^4}$$

For dP/dR to be zero, R must equal 1

QED!

2. Product Rule

The product rule states that the derivative of a product is

$$f(x)g(x)' = g(x)f'(x) + f(x)g'(x)$$

This may be applied if you think of the power equation as the product of R times $(R^2+2R+1)^{-1}$

$f(x)$ is R and therefore $f'(x)$ is 1

$g(x)$ is $(R^2+2R+1)^{-1}$ Because R^2+2R+1 may be written as $(R+1)^2$, this simplifies to $((R+1)^2)^{-1}$ or just $(R+1)^{-2}$ and therefore $g'(x)$ is $-2(R+1)^{-3}$

Plugging these values into the product formula yields

$$\frac{dP}{dR} = (R+1)^{-2} \cdot 1 - 2(R+1)^{-3} R$$

This is equivalent to

$$\frac{dP}{dR} = \frac{1}{(R+1)^2} - \frac{2R}{(R+1)^3}$$

Finding a common denominator and simplifying yields

$$\frac{dP}{dR} = \frac{R+1}{(R+1)^3} - \frac{2R}{(R+1)^3} = \frac{1-R}{(R+1)^3}$$

For dP/dR to be zero, R must equal 1

QED!

3. Chain Rule

The chain rule states that

$$\frac{dx}{dy} = \frac{dx}{du} \cdot \frac{du}{dy}$$

One way to apply the chain rule is first to rewrite the main equation to remove the numerator R. This effectively removes the issues of having a quotient or product. That is, divide through by R and

$$P = \frac{1}{R + 2 + \frac{1}{R}}$$

or in a somewhat more convenient form

$$P = (R + 2 + R^{-1})^{-1}$$

Using the chain rule the derivative of this is

$$\frac{dP}{dR} = -(R + 2 + R^{-1})^{-2} (1 - R^{-2})$$

or in “prettier” form

$$\frac{dP}{dR} = -\frac{1 - R^{-2}}{(R + 2 + R^{-1})^2}$$

Multiply through by R^2

$$\frac{dP}{dR} = \frac{1 - R^2}{(R^2 + 2R + 1)^2}$$

Which, for a really anal retentive sort of completeness, can be rewritten as

$$\frac{dP}{dR} = \frac{1 - R^2}{(R + 1)^4}$$

This is the precise result obtained using the quotient rule. As before, for dP/dR to be zero, R must equal 1

QED!

So, to reiterate, ***maximum load power will be obtained when the load resistance is equal to the voltage source's internal resistance.*** Further, at this point the system will be 50% efficient, meaning that half of the total generated power goes to the load while the other half is dissipated internally in the source.