# Fourier Analysis

or

## How to Build a Square Wave From a Bunch of Sines

Professor Fiore
Mohawk Valley Community College

The Fourier theorem, named after the French mathematician Jean-Baptiste Joseph Fourier, states that any repetitive waveform can be represented as a collection of sine and cosine waves of the proper amplitude and frequency. Alternately, it may be represented as a series of sine waves each with the proper amplitude, frequency and phase. This includes complex signals such as the human voice and musical instruments. Consequently, if a system is linear, by using superposition the response of a system to a complex wave may be understood in terms of its response to individual sine waves.

In this collection of waves, each component is known as a *partial* with the lowest frequency component known as the *fundamental*. All other partials are grouped together and referred to as *overtones*. “Regular” waveforms such as square waves and triangle waves feature a *harmonic overtone sequence* meaning that these overtones are integer multiples of the fundamental. As a shortcut, they are often referred to as just *harmonics*.

It might be hard to visualize initially, but both square waves and triangle waves are made up of a series of sines. The general equation for a square wave is:

$$v(t) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)2\pi f t$$

This says that a square wave of frequency *f* is made up of an infinite series of sines at odd integer multiples of *f*, with an inverse amplitude characteristic. For example, a 100 Hz square consists of a 100 Hz sine plus a 300 Hz sine at 1/3 amplitude plus a 500 Hz sine at 1/5 amplitude plus a 700 Hz sine at 1/7 amplitude and so on. A triangle wave is similar:

$$v(t) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)2\pi f t$$

Thus a triangle wave of frequency *f* is made up of an infinite series of cosines (sines with a 90 degree or one quarter cycle phase shift) at odd integer multiples of *f*, with an inverse square amplitude characteristic. For example, a 100 Hz triangle consists of a 100 Hz cosine plus a 300 Hz cosine at 1/9 amplitude plus a 500 Hz cosine at 1/25 amplitude plus a 700 Hz cosine at 1/49 amplitude and so on.

On the following page is a series of graphs showing the construction of a square wave. Along the left side are the fundamental and the first two harmonics at 100 Hz, 300 Hz and 500 Hz. At the top right is the fundamental added to the first harmonic and below it, the fundamental plus the first two harmonics. As more harmonics are added, the sides get steeper and the top/bottom start to flatten. They flatten because each additional harmonic partially cancels some of the peaks and valleys from the previous summation. This gives rise to a greater number of undulations with each undulation being smaller in vertical amplitude. Once a large number are added, the wave approaches a flat top and bottom with vertical sides. At the bottom right of the image is the result after adding approximately 100 harmonics. Notice the tight oscillations at the vertical edges. This is known as Gibbs phenomenon.